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Positive solutions of a prey–predator model with predator saturation and competition [☆]

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ABSTRACT

In this paper, we study the existence, multiplicity, bifurcation and stability of positive solutions to a prey–predator model with predator saturation and competition

$$\begin{cases} -\Delta u = u(a - u - bv f(u, v)), & x \in \Omega, \\ -\Delta v = v(c - v + duf(u, v)), & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases}$$

where

$$f(u, v) = \frac{1}{(1 + \alpha u)(1 + \beta v)},$$

and parameters are all positive constants, and u and v are the densities of the prey and predator, respectively.

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1. Introduction

Considering the destabilizing force of predator saturation and the stabilizing force of competition for prey, Bazykin [1] proposed the functional response

$$f(u, v) = \frac{1}{(1 + \alpha u)(1 + \beta v)}$$

in the prey–predator model instead of the classical Holling type II functional response. For this type functional response, the prey–predator model takes the form

$$\begin{cases} \frac{du}{dt} = u(a - u - bv f(u, v)), \\ \frac{dv}{dt} = v(c - v + duf(u, v)). \end{cases} \quad (1.1)$$

In this paper, we consider the positive solution of the boundary value problem of the following elliptic system corresponding to the system (1.1)

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$$\begin{cases} -\Delta u = u(a - u - bvf(u, v)), & x \in \Omega, \\ -\Delta v = v(c - v + duf(u, v)), & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases} \quad (1.2)$$

where the parameters are all positive constants, and Ω is a bounded domain in \mathbb{R}^N with smooth boundary $\partial\Omega$.

In papers [8,16], positive solutions were studied for the elliptic systems with ratio dependent functional response

$$\begin{cases} -\Delta u = u\left(a - u - \frac{bv}{u + mv}\right), & x \in \Omega, \\ -\Delta v = v\left(c - v + \frac{du}{u + mv}\right), & x \in \Omega, \\ k_1 \frac{\partial u}{\partial \nu} + u = k_2 \frac{\partial v}{\partial \nu} + v = 0, & x \in \partial\Omega, \end{cases} \quad (1.3)$$

and non-monotonic functional response

$$\begin{cases} -\Delta u = u\left(a - u - \frac{bv}{1 + mu + \beta u^2}\right), & x \in \Omega, \\ -\Delta v = v\left(c - v + \frac{du}{1 + mu + \beta u^2}\right), & x \in \Omega, \\ k_1 \frac{\partial u}{\partial \nu} + u = k_2 \frac{\partial v}{\partial \nu} + v = 0, & x \in \partial\Omega, \end{cases} \quad (1.4)$$

here ν is the outward unit normal vector of the boundary $\partial\Omega$, and k_1 and k_2 are non-negative constants.

Motivated by the papers [8,16], in the present paper, we study the positive solution of (1.2). In Section 2, we calculate the fixed point index by use of a well-known abstract theorem (Proposition 1). In Section 3, we apply the results obtained in Section 2 to study the existence of positive solutions. In Section 4, we discuss the stability and multiplicity of the positive solution when $\alpha \gg 1$, or $\beta \gg 1$, or $b \ll 1$. In Section 5, we investigate the bifurcation of positive solutions by using a and c as the main bifurcation parameters, respectively. And study the multiplicity and instability of positive solutions when d is sufficiently small. We should note that, the results of this paper are also true for the problem (1.2) instead of the homogeneous Dirichlet boundary conditions by the homogeneous Robin boundary conditions as that in (1.3).

For the related works on positive solutions of elliptic systems corresponding to prey–predator models, except the problems (1.3) and (1.4), the reader can refer to [2,5,6,9–15] and references therein.

Before ending this section, we give the a priori estimates of positive solutions of (1.2). It is well known that the following boundary value problem

$$\begin{cases} -\Delta u = u(a - u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \quad (1.5)$$

has a unique positive solution u_a if and only if $a > \lambda_1$, and u_a satisfies $\frac{\partial u_a}{\partial \nu}|_{\partial\Omega} < 0$. When $c > \lambda_1$, we denote by v_c the unique positive solution of the problem

$$\begin{cases} -\Delta v = v(c - v), & x \in \Omega, \\ v = 0, & x \in \partial\Omega. \end{cases}$$

The non-negative solutions $(u_a, 0)$ and $(0, v_c)$ are usually called *semi-trivial solutions* of the system (1.2).

Theorem 1. Any non-negative solution (u, v) of (1.2) has an a priori bounds:

$$u(x) \leq a, \quad v(x) \leq R := c + \frac{ad}{1 + a\alpha}.$$

Proof. Since u satisfies

$$-\Delta u = u(a - u - bvf(u, v)) \leq u(a - u),$$

then $u(x) \leq a$ by the maximum principle. Applying the maximum principle to the equation of v we get $v(x) \leq R$. \square

2. Calculations of the fixed point index

Let E be a Banach space. $W \subset E$ is called a *wedge* if W is a closed convex set and $\beta W \subset W$ for all $\beta \geq 0$. For $y \in W$, we define

$$W_y = \{x \in E: \exists r = r(x) > 0, \text{ s.t. } y + rx \in W\}, \quad S_y = \{x \in \overline{W}_y: -x \in \overline{W}_y\}.$$

We always assume that $E = \overline{W - W}$. Let $T : W_y \rightarrow W_y$ be a compact linear operator on E . We say that T has *property α* on \overline{W}_y if there exist $t \in (0, 1)$ and $w \in \overline{W}_y \setminus S_y$, such that $w - tTw \in S_y$.

For any $\delta > 0$ and $y \in W$, we denote $B_\delta^+(y) = B_\delta(y) \cap W$. Assume that $F : B_\delta^+(y) \rightarrow W$ is a compact operator and y is an isolated fixed point of F . If F is Fréchet differentiable at y , then the derivative $F'(y)$ has the property that $F'(y) : \overline{W}_y \rightarrow \overline{W}_y$. We denote by $\text{index}_W(F, y)$ the fixed point index of F at y relative to W .

Proposition 1. (See [4,11,15,17].) Assume that $I - F'(y)$ is invertible on \overline{W}_y . Then, we have

- (i) if $F'(y)$ has property α , then $\text{index}_W(F, y) = 0$,
- (ii) if $F'(y)$ does not have property α , then $\text{index}_W(F, y) = (-1)^\sigma$, where σ is the sum of multiplicities of all eigenvalues of $F'(y)$ which are greater than one.

For a linear operator A , we denote by $r(A)$ the spectral radius of A .

Proposition 2. (See [11,13].) Let $q(x) \in C^\alpha(\bar{\Omega})$ and M be a positive constant such that $M - q(x) > 0$ on $\bar{\Omega}$. Let $\lambda_1(q)$ be the first eigenvalue of the problem

$$\begin{cases} -\Delta u + q(x)u = \lambda u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \quad (2.1)$$

We have the following conclusions:

- (a) $\lambda_1(q) < 0 \Rightarrow r[(M - \Delta)^{-1}(M - q(x))] > 1$,
- (b) $\lambda_1(q) > 0 \Rightarrow r[(M - \Delta)^{-1}(M - q(x))] < 1$,
- (c) $\lambda_1(q) = 0 \Rightarrow r[(M - \Delta)^{-1}(M - q(x))] = 1$.

We introduce the following notations:

$$E = C_0^1(\bar{\Omega}) \times C_0^1(\bar{\Omega}), \quad \text{where } C_0^1(\bar{\Omega}) = \{w \in C^1(\bar{\Omega}) : w|_{\partial\Omega} = 0\},$$

$$W = K \times K, \quad \text{where } K = \{w \in C_0^1(\bar{\Omega}) : w(x) \geq 0\},$$

$$D = \{(u, v) \in W : u < a + 1, v < R + 1\}, \quad D^\circ = \text{int } D.$$

It is easy to prove that

- (i) $\overline{W}_{(0,0)} = K \times K$, $S_{(0,0)} = \{(0, 0)\}$,
- (ii) $\overline{W}_{(u_a,0)} = C_0^1(\bar{\Omega}) \times K$, $S_{(u_a,0)} = C_0^1(\bar{\Omega}) \times \{0\}$,
- (iii) $\overline{W}_{(0,v_c)} = K \times C_0^1(\bar{\Omega})$, $S_{(0,v_c)} = \{0\} \times C_0^1(\bar{\Omega})$.

From Theorem 1 we see that the non-negative solution of (1.2) must be in D . Take $M = 1 + a + (1 + b)(1 + R)$, then the functions

$$u(a - u - bvf(u, v)) + Mu \quad \text{and} \quad v(c - v + duf(u, v)) + Mv$$

are non-negative on \bar{D} . Define an operator $F : E \rightarrow E$ by

$$F(u, v) = (-\Delta + M)^{-1} \begin{pmatrix} u(a - u - bvf(u, v)) + Mu \\ v(c - v + duf(u, v)) + Mv \end{pmatrix},$$

then F is compact, and $F : D \rightarrow W$. Observe that (1.2) is equivalent to $(u, v) = F(u, v)$. For $t \in [0, 1]$, we define a positive and compact operator by

$$F_t(u, v) = (-\Delta + M)^{-1} \begin{pmatrix} tu(a - u - bvf(u, v)) + Mu \\ tv(c - v + duf(u, v)) + Mv \end{pmatrix},$$

then $F = F_1$.

Since we are only concerned with the non-negative solutions, in the sequel, the fixed point means that the non-negative one. Denote $\lambda_1(0) = \lambda_1$ for the simplicity, here $\lambda_1(q)$ is the first eigenvalue of the problem (2.1).

Lemma 1. Assume that $a > \lambda_1$. We have

- (i) $\deg_W(I - F, D) = 1$, here $\deg_W(I - F, D)$ is the degree of $I - F$ in D relative to W ,
- (ii) if $c \neq \lambda_1$, then $\text{index}_W(F, (0, 0)) = 0$,

- (iii) if $c > \lambda_1(-\frac{du_a}{1+\alpha u_a})$, then $\text{index}_W(F, (u_a, 0)) = 0$,
 (iv) if $c < \lambda_1(-\frac{du_a}{1+\alpha u_a})$, then $\text{index}_W(F, (u_a, 0)) = 1$.

Proof. (i) It is easy to see that F has no fixed point on ∂D . So, the degree $\deg_W(I - F, D)$ is well defined. For each t , a fixed point of F_t is a solution of the following problem

$$\begin{cases} -\Delta u = tu(a - u - bvf(u, v)), & x \in \Omega, \\ -\Delta v = tv(c - v + duf(u, v)), & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega. \end{cases} \quad (2.2)$$

As in Theorem 1, we see that the fixed point of F_t satisfies $u(x) \leq a$ and $v(x) \leq R$ on $\bar{\Omega}$ for each $t \in [0, 1]$, and so all fixed points of F_t are in D° . The degree $\deg_W(I - F_t, D)$ is independent of t . Therefore,

$$\deg_W(I - F, D) = \deg_W(I - F_1, D) = \deg_W(I - F_0, D).$$

Note that the problem (2.2) has only the trivial solution $(0, 0)$ when $t = 0$, we have $\deg_W(I - F_0, D) = \text{index}_W(F_0, (0, 0))$. Set

$$L = F'_0(0, 0) = (-\Delta + M)^{-1} \begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix}.$$

It is easy to check that $r(L) < 1$ by Proposition 2. This implies that $I - L$ is invertible on $\bar{W}_{(0,0)}$ and L does not have property α on $\bar{W}_{(0,0)}$. So, $\text{index}_W(F_0, (0, 0)) = 1$ by Proposition 1. The conclusion (i) is true.

(ii) Observe that $F(0, 0) = (0, 0)$ and F is compact on \bar{D} . Let $L = F'(0, 0)$. Then

$$L = (-\Delta + M)^{-1} \begin{pmatrix} a + M & 0 \\ 0 & c + M \end{pmatrix}.$$

Assume that $L(\xi, \eta) = (\xi, \eta)$ for some $(\xi, \eta) \in \bar{W}_{(0,0)}$. Then

$$\begin{cases} -\Delta \xi = a\xi, & x \in \Omega, \\ \xi = 0, & x \in \partial\Omega. \end{cases}$$

If $\xi > 0$, then $a = \lambda_1$, which is a contradiction, and so $\xi \equiv 0$. Similarly, $\eta \equiv 0$. Thus $I - L$ is invertible on $\bar{W}_{(0,0)}$.

Since $a > \lambda_1$, by Proposition 2 we see that $r_a \triangleq r[(-\Delta + M)^{-1}(a + M)] > 1$, and r_a is the principal eigenvalue of the operator $(-\Delta + M)^{-1}(a + M)$ with a corresponding eigenfunction $\phi > 0$. Set $t_0 = r_a^{-1}$, then $0 < t_0 < 1$ and $(I - t_0 L)(\phi, 0) = (0, 0) \in S_{(0,0)}$. This shows that L has property α . Thus, $\text{index}_W(F, (0, 0)) = 0$ by Proposition 1.

(iii) Set $L = F'(u_a, 0)$. Then we have

$$L = (-\Delta + M)^{-1} \begin{pmatrix} a - 2u_a + M & -\frac{bu_a}{1+\alpha u_a} \\ 0 & c + \frac{du_a}{1+\alpha u_a} + M \end{pmatrix}.$$

Assume that $L(\xi, \eta) = (\xi, \eta)$ for some $(\xi, \eta) \in \bar{W}_{(u_a,0)}$. Then

$$\begin{cases} -\Delta \xi + (2u_a - a)\xi = -\frac{bu_a}{1+\alpha u_a}\eta, & x \in \Omega, \\ -\Delta \eta - \frac{du_a}{1+\alpha u_a}\eta = c\eta, & x \in \Omega, \\ \xi = \eta = 0, & x \in \partial\Omega. \end{cases} \quad (2.3)$$

Taking account of $\eta \in K$, it follows from the second equation of (2.3) that $c = \lambda_1(-\frac{du_a}{1+\alpha u_a})$ if $\eta \neq 0$ (Proposition 2). Since $c > \lambda_1(-\frac{du_a}{1+\alpha u_a})$, we have $\eta \equiv 0$. If $\xi \neq 0$, then 0 is an eigenvalue of the following problem

$$\begin{cases} -\Delta \phi + (2u_a - a)\phi = \lambda \phi, & x \in \Omega, \\ \phi = 0, & x \in \partial\Omega. \end{cases}$$

Thus $\lambda_1(2u_a - a) \leq 0$. Using the comparison property of eigenvalues, it yields $\lambda_1(2u_a - a) > \lambda_1(u_a - a) = 0$. We get a contradiction. Therefore, $(\xi, \eta) = (0, 0)$, i.e. $I - L$ is invertible on $\bar{W}_{(u_a,0)}$.

We claim that L has property α on $\bar{W}_{(u_a,0)}$. In fact, set

$$\mathcal{A} \triangleq (-\Delta + M)^{-1} \left(c + \frac{du_a}{1+\alpha u_a} + M \right).$$

Since $c > \lambda_1(-\frac{du_a}{1+\alpha u_a})$, by Proposition 2 we have that $r_c \triangleq r(\mathcal{A}) > 1$ is an eigenvalue of \mathcal{A} with a corresponding eigenfunction $\phi > 0$. Set $t_0 := r_c^{-1}$. Then $t_0 \in (0, 1)$, $(0, \phi) \in \overline{W}_{(u_a, 0)} \setminus S_{(u_a, 0)}$, and

$$(I - t_0 L) \begin{pmatrix} 0 \\ \phi \end{pmatrix} = \begin{pmatrix} (-\Delta + M)^{-1} \frac{t_0 b u_a \phi}{1 + \alpha u_a} \\ \phi - t_0 (-\Delta + M)^{-1} (c + \frac{du_a}{1 + \alpha u_a} + M) \phi \end{pmatrix} = \begin{pmatrix} (-\Delta + M)^{-1} \frac{t_0 b u_a \phi}{1 + \alpha u_a} \\ 0 \end{pmatrix} \in S_{(u_a, 0)}.$$

This proves that L has property α . Therefore, $\text{index}_W(F, (u_a, 0)) = 0$ by Proposition 1.

(iv) Similar as (iii), $I - L$ is invertible on $\overline{W}_{(u_a, 0)}$. Now we prove that L does not have property α on $\overline{W}_{(u_a, 0)}$. Since $c < \lambda_1(-\frac{du_a}{1+\alpha u_a})$, we have $r(\mathcal{A}) < 1$. On the contrary we suppose that L has property α on $\overline{W}_{(u_a, 0)}$. Then there exist $0 < t < 1$ and $(\phi_1, \phi_2) \in \overline{W}_{(u_a, 0)} \setminus S_{(u_a, 0)}$, such that $(I - tL)(\phi_1, \phi_2) \in S_{(u_a, 0)}$. So

$$\phi_2 - t(-\Delta + M)^{-1} \left(c + \frac{du_a}{1 + \alpha u_a} + M \right) \phi_2 = 0.$$

Since $\phi_2 \in K \setminus \{0\}$, it follows that $1/t > 1$ is an eigenvalue of the operator \mathcal{A} , which is a contradiction to $r(\mathcal{A}) < 1$. So L does not have property α on $\overline{W}_{(u_a, 0)}$. By Proposition 1,

$$\text{index}_W(F, (u_a, 0)) = (-1)^\sigma,$$

where σ is the sum of the multiplicities of all real eigenvalues of L which are greater than 1.

Assume that $1/\mu > 1$ is an eigenvalue of L with a corresponding eigenfunction (ξ, η) . Then

$$(-\Delta + M)^{-1} \begin{pmatrix} (a - 2u_a + M)\xi - \frac{bu_a}{1 + \alpha u_a} \eta \\ (c + \frac{du_a}{1 + \alpha u_a} + M)\eta \end{pmatrix} = \frac{1}{\mu} \begin{pmatrix} \xi \\ \eta \end{pmatrix},$$

equivalently,

$$\begin{cases} -\Delta \xi + M \xi = \mu \left((a - 2u_a + M)\xi - \frac{bu_a}{1 + \alpha u_a} \eta \right), & x \in \Omega, \\ -\Delta \eta + M \eta = \mu \left(c + \frac{du_a}{1 + \alpha u_a} + M \right) \eta, & x \in \Omega, \\ \xi = \eta = 0, & x \in \partial \Omega. \end{cases} \quad (2.4)$$

If $\eta \neq 0$, it follows from the second equation of (2.4) that

$$0 = \lambda_1 \left(M(1 - \mu) - \mu \left(c + \frac{du_a}{1 + \alpha u_a} \right) \right) > \lambda_1 \left(- \left(c + \frac{du_a}{1 + \alpha u_a} \right) \right) = -c + \lambda_1 \left(-\frac{du_a}{1 + \alpha u_a} \right),$$

which is a contradiction to $c < \lambda_1(-\frac{du_a}{1+\alpha u_a})$, and so $\eta \equiv 0$. Thus $\xi \neq 0$. Observe that $u_a \leq a$, it follows from the first equation of (2.4) that

$$0 = \lambda_1 (M(1 - \mu) - \mu(a - 2u_a)) > \lambda_1 (-\mu(a - u_a)) \geq \lambda_1 (-(a - u_a)) = 0.$$

This contradiction shows that L has no eigenvalues being greater than 1. Consequently, $\sigma = 0$. Hence $\text{index}_W(F, (u_a, 0)) = 1$. \square

Similar to the above, we can prove the following lemma.

Lemma 2. Assume that $c > \lambda_1$. We have

- (i) if $a > \lambda_1(\frac{bv_c}{1+\beta v_c})$, then $\text{index}_W(F, (0, v_c)) = 0$,
- (ii) if $a < \lambda_1(\frac{bv_c}{1+\beta v_c})$, then $\text{index}_W(F, (0, v_c)) = 1$.

3. Existence of positive solutions

Theorem 2.

- (i) If $c > \lambda_1$ and $a > \lambda_1(\frac{bv_c}{1+\beta v_c})$, then (1.2) has at least one positive solution.
- (ii) If the problem (1.2) has a positive solution (u, v) , then $a > \lambda_1$ and $u < u_a$.
- (iii) Assume that $c < \lambda_1$. Then (1.2) has positive solution if and only if $a > \lambda_1$ and $c > \lambda_1(-\frac{du_a}{1+\alpha u_a})$.

Proof. (i) By Lemmas 1 and 2, we have

$$\deg_W(I - F, D) - \text{index}_W(F, (0, 0)) - \text{index}_W(F, (u_a, 0)) - \text{index}_W(F, (0, v_c)) = 1.$$

So (1.2) has at least one positive solution.

The conclusion of (ii) is obvious.

(iii) We first prove the sufficiency. Since $c < \lambda_1$, the problem (1.2) has no solution taking the form $(0, v)$ with $v(x) > 0$. If $a > \lambda_1$ and $c > \lambda_1(-\frac{du_a}{1+\alpha u_a})$, note that $c < \lambda_1$, by Lemma 1 we have

$$\deg_W(I - F, D) - \text{index}_W(F, (0, 0)) - \text{index}_W(F, (u_a, 0)) = 1.$$

Hence (1.2) has at least one positive solution.

Conversely, assume that (\bar{u}, \bar{v}) is a positive solution of (1.2). Then $a > \lambda_1$ and $\bar{u} < u_a$ by (ii). Since (\bar{u}, \bar{v}) satisfies

$$\begin{cases} -\Delta \bar{v} = \bar{v}(c - \bar{v} + d\bar{u}f(\bar{u}, \bar{v})), & x \in \Omega, \\ \bar{v} = 0, & x \in \partial\Omega, \end{cases}$$

one has

$$0 = \lambda_1(\bar{v} - c - d\bar{u}f(\bar{u}, \bar{v})) > \lambda_1\left(-c - \frac{du_a}{1+\alpha u_a}\right). \quad \square$$

Theorem 3. If one of the following holds, then (1.2) has no positive solutions:

- (i) $a \leq c$ and $b \geq (1 + \alpha a)(1 + \beta R)$;
- (ii) $b < (1 + \alpha a)(1 + \beta R)$ and $c - a \geq [1 - bf(a, R)]R$.

Proof. Suppose that there is a positive solution (u, v) of (1.2). For the case (i), in view of Theorem 1 we have

$$\begin{aligned} 0 &= \lambda_1(u - a + bvf(u, v)) > \lambda_1(v - c - duf(u, v) + (bf(u, v) - 1)v) \\ &> \lambda_1(v - c - duf(u, v) + (bf(a, R) - 1)v) \geq \lambda_1(v - c - duf(u, v)) = 0, \end{aligned}$$

which is a contradiction.

For the case (ii), the assumption implies that $a < c$. Similar to the above, we have

$$\begin{aligned} 0 &= \lambda_1(u - a + bvf(u, v)) > \lambda_1(v - c - duf(u, v) + c - a + (bf(u, v) - 1)v) \\ &> \lambda_1(v - c - duf(u, v) + c - a + (bf(a, R) - 1)R) \geq \lambda_1(v - c - duf(u, v)) = 0, \end{aligned}$$

which derives a contradiction. \square

4. Stability and multiplicity of positive solutions

In this section, the stability and multiplicity of the positive solution when $\alpha \gg 1$, or $\beta \gg 1$, or $b \ll 1$ will be studied. Since β is upside with α , we only discuss the case $\alpha \gg 1$, or $b \ll 1$. We first discuss the case of $\alpha \gg 1$. For this aim, an asymptotic result is given firstly.

Lemma 3. Assume that $a > \lambda_1$ and $c > \lambda_1$. For any given small $\varepsilon > 0$ satisfying $\varepsilon < a - \lambda_1$, there exists $\bar{\alpha}(\varepsilon) > 0$, such that when $\alpha \geq \bar{\alpha}(\varepsilon)$, (1.2) has at least one positive solution (u, v) and satisfies

$$u_{a-\varepsilon} \leq u \leq u_a, \quad v_c \leq v \leq v_{c+\varepsilon}. \quad (4.1)$$

Proof. Let $\underline{u} = (\underline{u}, \underline{v}) = (u_{a-\varepsilon}, v_c)$ and $\bar{u} = (\bar{u}, \bar{v}) = (u_a, v_{c+\varepsilon})$. It is easy to check that functions $u(a - u - bvf(u, v))$ and $v(c - v + duf(u, v))$ are Lipschitz continuous in (\underline{u}, \bar{u}) . If we can prove that \bar{u} and \underline{u} are the upper and lower solutions of (1.2), respectively, then the problem (1.2) has at least one positive solution (u, v) and satisfies (4.1).

To do this, it suffices to require that the following hold:

$$\Delta \bar{u} + \bar{u}(a - \bar{u} - b\bar{v}f(\bar{u}, \bar{v})) \leq 0, \quad (4.2)$$

$$\Delta \underline{u} + \underline{u}(a - \underline{u} - b\bar{v}f(\underline{u}, \bar{v})) \geq 0, \quad (4.3)$$

$$\Delta \bar{v} + \bar{v}(c - \bar{v} + d\bar{u}f(\bar{u}, \bar{v})) \leq 0, \quad (4.4)$$

$$\Delta \underline{v} + \underline{v}(c - \underline{v} + d\bar{u}f(\underline{u}, \bar{v})) \geq 0. \quad (4.5)$$

Inequalities (4.2) and (4.5) are obvious. Next we check (4.3) and (4.4). By the direct calculation,

$$\Delta \underline{u} + \underline{u}(a - \underline{u} - b\bar{v}f(\underline{u}, \bar{v})) = u_{a-\varepsilon}(\varepsilon - bv_{c+\varepsilon}f(u_{a-\varepsilon}, v_{c+\varepsilon})), \quad (4.6)$$

$$\Delta \bar{v} + \bar{v}(c - \bar{v} + d\bar{u}f(\bar{u}, \bar{v})) = v_{c+\varepsilon}(-\varepsilon + du_{a-\varepsilon}f(u_{a-\varepsilon}, v_{c+\varepsilon})). \quad (4.7)$$

Since $u_a(x) = 0$ and $v_{c+\varepsilon}(x) = 0$ on $\partial\Omega$. The right-hand side of (4.6) is positive and that of (4.7) is negative near $\partial\Omega$. Since

$$bv_{c+\varepsilon}f(u_{a-\varepsilon}, v_{c+\varepsilon}) \rightarrow 0, \quad du_{a-\varepsilon}f(u_{a-\varepsilon}, v_{c+\varepsilon}) \rightarrow 0$$

uniformly on any compact subset of Ω as $\alpha \rightarrow \infty$, the right-hand side of (4.6) is positive and that of (4.7) is negative on any fixed compact subset of Ω when $\alpha \gg 1$. Therefore, inequalities (4.3) and (4.4) hold when $\alpha \gg 1$. \square

Theorem 4. Assume that $a > \lambda_1$ and $c > \lambda_1$. Then, as $\alpha \gg 1$, the problem (1.2) has at least one non-degenerate and linearly stable positive solution (u, v) .

Proof. Choose $0 < \varepsilon_i \rightarrow 0$. By Lemma 3, there exists $\bar{\alpha}(\varepsilon_i) \gg 1$, such that when $\alpha \geq \bar{\alpha}(\varepsilon_i)$, the problem (1.2) has a positive solution, denoted by (u_α, v_α) , and satisfies

$$u_{a-\varepsilon_i} \leq u_\alpha \leq u_a, \quad v_c \leq v_\alpha \leq v_{c+\varepsilon_i}. \quad (4.8)$$

We shall prove that, when $i \gg 1$, such a positive solution (u_α, v_α) is non-degenerate and linearly stable. That is, to prove that the corresponding linearized eigenvalue problem for (1.2) at (u_α, v_α) has no eigenvalue μ satisfying $\operatorname{Re}(\mu) \leq 0$. If this is not true, we can find $\alpha_i \rightarrow \infty$, μ_i with $\operatorname{Re}(\mu_i) \leq 0$ and $(\xi_i, \eta_i) \neq (0, 0)$ satisfying $\|\xi_i\|_2^2 + \|\eta_i\|_2^2 = 1$, such that

$$\begin{cases} -\Delta \xi_i - (a - 2u_i - f_i)\xi_i + f_i^* \eta_i = \mu_i \xi_i, & x \in \Omega, \\ -\Delta \eta_i - g_i \xi_i - (c - 2v_i + g_i^*)\eta_i = \mu_i \eta_i, & x \in \Omega, \\ \xi_i = \eta_i = 0, & x \in \partial\Omega, \end{cases} \quad (4.9)$$

where $(u_i, v_i) = (u_{\alpha_i}, v_{\alpha_i})$, and

$$\begin{aligned} f_i &= \frac{bv_i}{(1 + \alpha_i u_i)^2 (1 + \beta v_i)}, & f_i^* &= \frac{bu_i}{(1 + \alpha_i u_i)(1 + \beta v_i)^2}, \\ g_i &= \frac{dv_i}{(1 + \alpha_i u_i)^2 (1 + \beta v_i)}, & g_i^* &= \frac{du_i}{(1 + \alpha_i u_i)(1 + \beta v_i)^2}. \end{aligned}$$

Multiplying the equations of (4.9) by $\bar{\xi}_i$ and $\bar{\eta}_i$ respectively, and integrating them over Ω , and adding the results, we have

$$\mu_i = \int_{\Omega} (|\nabla \xi_i|^2 + |\nabla \eta_i|^2) dx + \int_{\Omega} [|\xi_i|^2 (f_i + 2u_i - a) + f_i^* \eta_i \bar{\xi}_i] dx - \int_{\Omega} [g_i \xi_i \bar{\eta}_i + |\eta_i|^2 (c - 2v_i + g_i^*)] dx, \quad (4.10)$$

where $\bar{\xi}_i$ and $\bar{\eta}_i$ are the complex conjugates of ξ_i and η_i , respectively. Since u_i and v_i are bounded (Theorem 1), and $\operatorname{Re}(\mu_i) \leq 0$, $\|\xi_i\|_2^2 + \|\eta_i\|_2^2 = 1$, it follows from (4.10) that $\operatorname{Re}(\mu_i)$ and $\operatorname{Im}(\mu_i)$ are bounded. So $\{\mu_i\}$ is bounded. Without loss of generality we assume that $\mu_i \rightarrow \mu$ with $\operatorname{Re}(\mu) \leq 0$. Using the L^p theory to (4.9) we see that ξ_i and η_i are bounded in $W_p^2(\Omega)$ for all $p > n$. There exists a subsequence, denoted by itself, such that $\xi_i \rightarrow \xi$, $\eta_i \rightarrow \eta$ in $W_p^1(\Omega)$.

In view of $\varepsilon_i \rightarrow 0$ and the estimate (4.8), taking the limit in (4.9) we have that, in the weak sense, (ξ, η) satisfies

$$\begin{cases} -\Delta \xi - \xi(a - 2u_a) = \mu \xi, & x \in \Omega, \\ -\Delta \eta - \eta(c - 2v_c) = \mu \eta, & x \in \Omega, \\ \xi = \eta = 0, & x \in \partial\Omega. \end{cases} \quad (4.11)$$

Since $\xi, \eta \in W_p^1(\Omega) \hookrightarrow C^\alpha(\bar{\Omega})$, (4.11) is satisfied in the classical sense by the regularity theory of elliptic equations. Since the problem (4.11) is symmetric, μ is real, and in turn $\mu \leq 0$.

If $\xi \neq 0$, then μ is an eigenvalue of the problem

$$\begin{cases} -\Delta \phi + (2u_a - a)\phi = \mu \phi, & x \in \Omega, \\ \phi = 0, & x \in \partial\Omega. \end{cases}$$

This implies $0 \geq \mu \geq \lambda_1(2u_a - a)$. As $\lambda_1(2u_a - a) > \lambda_1(u_a - a) = 0$, we get a contradiction. So $\xi \equiv 0$. Similarly, $\eta \equiv 0$. It is a contradiction since $\|\xi\|_2^2 + \|\eta\|_2^2 = 1$. \square

Theorem 5. Assume that $c > \lambda_1$ and $\lambda_1 < a < \lambda_1(\frac{bv_c}{1+\beta v_c})$. Then (1.2) has at least two positive solutions when $\alpha \gg 1$.

Proof. By Theorem 4 we know that, when $\alpha \gg 1$, the problem (1.2) has at least one non-degenerate and linearly stable positive solution (\tilde{u}, \tilde{v}) . So $I - F'(\tilde{u}, \tilde{v})$ is invertible on $\overline{W}_{(\tilde{u}, \tilde{v})}$ and $F'(\tilde{u}, \tilde{v})$ has no real eigenvalue being greater than 1. Note that $\overline{W}_{(\tilde{u}, \tilde{v})} = [C_0^1(\overline{\Omega})]^2 = S_{(\tilde{u}, \tilde{v})}$, it is easy to see that $F'(\tilde{u}, \tilde{v})$ does not have property α . By Proposition 1, $\text{index}_W(F, (\tilde{u}, \tilde{v})) = 1$. If the problem (1.2) has only such one positive solution (\tilde{u}, \tilde{v}) , applying Lemmas 1 and 2, we have

$$\begin{aligned} 1 &= \deg_W(I - F, D) = \text{index}_W(F, (0, 0)) + \text{index}_W(F, (u_a, 0)) + \text{index}_W(F, (0, v_c)) + \text{index}_W(F, (\tilde{u}, \tilde{v})) \\ &= 0 + 0 + 1 + 1, \end{aligned}$$

which is a contradiction. \square

Next, we discuss the stability of the positive solution as $b \rightarrow 0^+$. If the problem (1.2) has positive solutions, then $a > \lambda_1$ by (ii) of Theorem 2. When $c \geq \lambda_1$, we have $c > \lambda_1(-\frac{du_a}{1+\alpha u_a})$. When $c < \lambda_1$, we have $c > \lambda_1(-\frac{du_a}{1+\alpha u_a})$ by (iii) of Theorem 2. In a word, if the problem (1.2) has a positive solution then $a > \lambda_1$ and $c > \lambda_1(-\frac{du_a}{1+\alpha u_a})$. Note that the function $h(x, v) := c - v + du_a(x)f(u_a(x), v)$ is monotone decreasing for $v > 0$, and $h(x, v) < 0$ when $v \gg 1$, in view of $c > \lambda_1(-\frac{du_a}{1+\alpha u_a})$ we see that the problem

$$\begin{cases} -\Delta v = v(c - v + du_a f(u_a, v)), & x \in \Omega, \\ v = 0, & x \in \partial\Omega, \end{cases} \quad (4.12)$$

has a unique positive solution, denoted by \hat{v} .

Theorem 6. If $a > \lambda_1$, then the positive solution (u, v) of (1.2) (if it exists) converges to (u_a, \hat{v}) as $b \rightarrow 0^+$.

Proof. It is easy to see that the compact operator $F(u, v)$ converges to the compact operator

$$\tilde{F}(u, v) = (-\Delta + M)^{-1} \begin{pmatrix} u(a - u) + Mu \\ v(c - v + du f(u, v)) + Mv \end{pmatrix}$$

in D° as $b \rightarrow 0^+$. So the fixed points of $F(u, v)$ converge to the fixed points of $\tilde{F}(u, v)$ in D° as $b \rightarrow 0^+$. Since $\tilde{F}(u, v)$ has a unique fixed point (u_a, \hat{v}) in D° , the result follows. \square

Theorem 7. If $a > \lambda_1$ and $c > \lambda_1(-\frac{du_a}{1+\alpha u_a})$, then $\lambda_1(2\hat{v} - c - \frac{du_a}{(1+\alpha u_a)(1+\beta \hat{v})^2}) > 0$.

Proof. Since \hat{v} is a positive solution of (4.12), we see that $\lambda_1(\hat{v} - c - du_a f(u_a, \hat{v})) = 0$. Let $g(x, v) = c - v + du_a f(u_a, v)$, then

$$g_v = -1 - \frac{\beta du_a}{(1 + \alpha u_a)(1 + \beta v)^2} < 0.$$

It follows that

$$\begin{aligned} 0 &= \lambda_1(-g(x, \hat{v})) < \lambda_1(-g(x, \hat{v}) - \hat{v} g_v(x, \hat{v})) = \lambda_1\left(\hat{v} - c - \frac{du_a}{(1 + \alpha u_a)(1 + \beta \hat{v})} + \hat{v} + \frac{\beta du_a \hat{v}}{(1 + \alpha u_a)(1 + \beta \hat{v})^2}\right) \\ &= \lambda_1\left(2\hat{v} - c - \frac{du_a}{(1 + \alpha u_a)(1 + \beta \hat{v})^2}\right). \quad \square \end{aligned}$$

Theorem 8. If $a > \lambda_1$, then there exists a positive constant b^* such that (1.2) has at most one positive solution when $b \leq b^*$. Moreover, the positive solution (if it exists) is non-degenerate and linearly stable.

Proof. When $b \ll 1$. If (1.2) has a positive solution, the uniqueness is an immediate consequence of the *Implicit Function Theorem* using b as the main parameter and using Theorems 6 and 7. Similar to the proof of Theorem 4(ii), the positive solution (if it exists) is non-degenerate and linearly stable. \square

5. Bifurcation, instability and multiplicity of positive solutions

In the last section, we have discussed the instability and multiplicity of the positive solution when $\alpha \gg 1$, or $\beta \gg 1$, or $b \ll 1$. In this section, we will investigate the bifurcation of positive solutions by using a and c as the main bifurcation parameters, respectively. And study the multiplicity and instability of positive solutions when d is sufficiently small.

Theorem 9.

- (i) Assume that $c > \lambda_1$ and denote $\tilde{a} = \lambda_1(\frac{bv_c}{1+\beta v_c})$. Then the point $(0, v_c, \tilde{a})$ is a bifurcation point of the positive solution of (1.2). Moreover, when $0 < s \ll 1$, the bifurcating positive solution $(u(s), v(s), a(s))$ from $(0, v_c, \tilde{a})$ takes the form

$$\begin{cases} u(s) = s\Phi + O(s^2), \\ v(s) = v_c + s\Psi_d + O(s^2), \\ a(s) = \tilde{a} + a_1s + O(s^2), \end{cases} \quad (5.1)$$

where $\Psi_d = d(-\Delta - c + 2v_c)^{-1}(\frac{v_c}{1+\beta v_c}\Phi)$, and Φ is the positive eigenfunction corresponding to \tilde{a} with $\int_{\Omega} \Phi^2 dx = 1$. By substituting $(u(s), v(s), a(s))$ into the first equation of (1.2) we obtain

$$a_1 = \frac{\int_{\Omega} [(1 + \beta v_c)^2 \Phi^2 - \alpha b v_c (1 + \beta v_c) \Phi^2 + b \Psi_d \Phi] dx}{\int_{\Omega} (1 + \beta v_c)^2 \Phi^2 dx}. \quad (5.2)$$

- (ii) Assume that $a > \lambda_1$ and denote $\tilde{c} = \lambda_1(-\frac{du_a}{1+\alpha u_a})$. Then the point $(u_a, 0, \tilde{c})$ is a bifurcation point of the positive solution of (1.2). Moreover, when $0 < s \ll 1$, the bifurcating positive solution $(u(s), v(s), c(s))$ from $(u_a, 0, \tilde{c})$ has the form

$$\begin{cases} u(s) = u_a + s\hat{\Psi} + O(s^2), \\ v(s) = s\Phi + O(s^2), \\ c(s) = \tilde{c} + c_1s + O(s^2), \end{cases}$$

where $\hat{\Psi} = b(-\Delta - a + 2u_a)^{-1}(-\frac{v_c}{1+\beta v_c}\hat{\Phi})$, and $\hat{\Phi}$ is the positive eigenfunction corresponding to \tilde{c} with $\int_{\Omega} \hat{\Phi}^2 dx = 1$. By substituting $(u(s), v(s), a(s))$ into the second equation of (1.2) we have

$$c_1 := \frac{\int_{\Omega} [(1 + \alpha u_a)^2 \hat{\Phi}^2 + \beta d u_a (1 + \alpha u_a) \hat{\Phi}^2 - d \hat{\Psi} \hat{\Phi}] dx}{\int_{\Omega} (1 + \alpha u_a)^2 \hat{\Phi}^2 dx}.$$

Proof. We only prove the conclusion (i). Define a mapping $\mathbf{F} : E \times \mathbb{R} \rightarrow E$ by

$$\mathbf{F}(u, v, a) := \begin{pmatrix} \Delta u + u(a - u - bvf(u, v)) \\ \Delta v + v(c - v + duf(u, v)) \end{pmatrix}.$$

By the simple calculation, for each $(\xi, \eta) \in E$,

$$\mathbf{F}_{(u,v)}(u, v, a) \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \Delta \xi + \xi(a - 2u - \frac{bvf(u,v)}{1+\alpha u}) - \frac{bvf(u,v)}{1+\beta v} \eta \\ \Delta \eta + \frac{dvf(u,v)}{1+\alpha u} \xi + (c - 2v + \frac{duf(u,v)}{1+\beta v}) \eta \end{pmatrix},$$

and so

$$\mathbf{F}_{(u,v)}(0, v_c, \tilde{a}) \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \Delta \xi + \xi(\tilde{a} - \frac{bv_c}{1+\beta v_c}) \\ \Delta \eta + \frac{dv_c}{1+\beta v_c} \xi + (c - 2v_c) \eta \end{pmatrix}.$$

Claim 1. $\dim(\text{Ker } \mathbf{F}_{(u,v)}(0, v_c, \tilde{a})) = 1$ and $\text{Ker } \mathbf{F}_{(u,v)}(0, v_c, \tilde{a}) = \text{span}\{(\Phi, \Psi_d)\}$.

Suppose that $\mathbf{F}_{(u,v)}(0, v_c, \tilde{a})(\xi, \eta) = (0, 0)$ for some $(\xi, \eta) \in E$, then we have

$$\begin{cases} -\Delta \xi - \xi \left(\tilde{a} - \frac{bv_c}{1+\beta v_c} \right) = 0, & x \in \Omega, \\ -\Delta \eta - \frac{dv_c}{1+\beta v_c} \xi - (c - 2v_c) \eta = 0, & x \in \Omega, \\ \xi = \eta = 0, & x \in \partial \Omega. \end{cases}$$

It follows that $\xi = \text{span}\{\Phi\}$ since $\tilde{a} = \lambda_1(\frac{bv_c}{1+\beta v_c})$, and so $\eta = \text{span}\{\Psi_d\}$.

Claim 2. $\text{Codim}(\mathcal{RF}_{(u,v)}(0, v_c, \tilde{a})) = 1$.

Assume that $(\tilde{\xi}, \tilde{\eta}) \in \mathcal{RF}_{(u,v)}(0, v_c, \tilde{a})$. Then there is $(\xi, \eta) \in E$ such that

$$\begin{cases} \Delta \xi + \xi \left(\tilde{a} - \frac{bv_c}{1+\beta v_c} \right) = \tilde{\xi}, & x \in \Omega, \\ \Delta \eta + \frac{dv_c}{1+\beta v_c} \xi - (c - 2v_c) \eta = \tilde{\eta}, & x \in \Omega, \\ \xi = \eta = 0, & x \in \partial \Omega. \end{cases} \quad (5.3)$$

It follows that $\int_{\Omega} \tilde{\Phi} \tilde{\xi} dx = 0$ since $\tilde{\Phi}$ is the eigenfunction corresponding to $\tilde{a} = \lambda_1(\frac{bv_c}{1+\beta v_c})$. So $(\tilde{\xi}, \tilde{\eta})$ is orthogonal to $(\tilde{\Phi}, 0)$.

Conversely, if $(\tilde{\xi}, \tilde{\eta})$ is orthogonal to $(\tilde{\Phi}, 0)$, then the first equation of (5.3) has a unique solution ξ . Therefore, the second equation of (5.3) has a unique solution η since $\Delta - c + 2v_c$ is invertible. Thus $(\tilde{\xi}, \tilde{\eta}) \in \mathcal{RF}_{(u,v)}(0, v_c, \tilde{a})$. And so $\text{Codim}(\mathcal{RF}_{(u,v)}(0, v_c, \tilde{a})) = 1$.

Claim 3. $\mathbf{F}_{(u,v),a}(0, v_c, \tilde{a})(\Phi, \Psi_d) \notin \mathcal{RF}_{(u,v)}(0, v_c, \tilde{a})$.

Since $\mathcal{RF}_{(u,v)}(0, v_c, \tilde{a})$ is orthogonal to $(\Phi, 0)$ and $\mathbf{F}_{(u,v),a}(0, v_c, \tilde{a})(\Phi, \Psi_d) = (\Phi, 0)$, we have

$$\mathbf{F}_{(u,v),a}(0, v_c, \tilde{a})(\Phi, \Psi_d) \notin \mathcal{RF}_{(u,v)}(0, v_c, \tilde{a}).$$

Finally, applying the bifurcation theorem [3], we conclude the desired results. \square

From the above proof we see that $(0, v_c, \tilde{a})$ is the bifurcation point of positive solutions for the small $d > 0$. The following theorem asserts that the bifurcating positive solution from $(0, v_c, \tilde{a})$ is unstable provided that d is small.

Theorem 10. Assume that $c > \lambda_1$ and $\int_{\Omega} \Phi^3 (1 - \frac{\alpha bv_c}{1+\beta v_c}) dx \neq 0$. Then, as $d \ll 1$, the local bifurcation of positive solution $(u(s), v(s))$ bifurcating from $(0, v_c, \tilde{a})$ is non-degenerate. Moreover, $(u(s), v(s))$ is unstable if $\int_{\Omega} \Phi^3 (1 - \frac{\alpha bv_c}{1+\beta v_c}) dx < 0$, and stable if $\int_{\Omega} \Phi^3 (1 - \frac{\alpha bv_c}{1+\beta v_c}) dx > 0$. If in addition, the constant a_1 determined by (5.2) is negative. Then, as $d \ll 1$, $a < \tilde{a}$ and nears \tilde{a} , the problem (1.2) has at least two positive solutions.

Proof. Step 1. Choose sequences $\{s_i\}_{i=1}^{\infty}$ and $\{d_i\}_{i=1}^{\infty}$ with $s_i, d_i \rightarrow 0^+$. Denote $a_i = a(s_i)$ and $(u_i, v_i) = (u(s_i), v(s_i))$. Then the corresponding linearized problem at (u_i, v_i) can be written as

$$L_i \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \mu \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \quad L_i = \begin{pmatrix} M_i^{11} & M_i^{12} \\ M_i^{21} & M_i^{22} \end{pmatrix},$$

where

$$\begin{aligned} M_i^{11} &= -\Delta - \left(a_i - 2u_i - \frac{bv_i f(u_i, v_i)}{1 + \alpha u_i} \right), & M_i^{12} &= \frac{bu_i f(u_i, v_i)}{1 + \beta v_i}, \\ M_i^{21} &= -\frac{d_i v_i f(u_i, v_i)}{1 + \alpha u_i}, & M_i^{22} &= -\Delta - \left(c - 2v_i + \frac{d_i u_i f(u_i, v_i)}{1 + \beta v_i} \right). \end{aligned}$$

It is obvious that, as $i \rightarrow \infty$,

$$L_i \rightarrow L_0 = \begin{pmatrix} -\Delta + (\frac{bv_c}{1+\beta v_c} - \tilde{a}) & 0 \\ 0 & -\Delta - (c - 2v_c) \end{pmatrix}.$$

Since $\tilde{a} = \lambda_1(\frac{bv_c}{1+\beta v_c})$, it is easy to see the 0 is the first eigenvalue of the operator $-\Delta + \frac{bv_c}{1+\beta v_c} - \tilde{a}$. On the other hand, since $c = \lambda_1(v_c) < \lambda_1(2v_c)$, the first eigenvalue of $-\Delta - (c - 2v_c)$ is positive. Therefore, 0 is the first eigenvalue of L_0 with the corresponding eigenfunction $(\Phi, 0)$. Moreover, all the other eigenvalues of L_0 are positive and apart from 0. By the perturbation theory of linear operators (see [7, Chapter IV, Section 3.5]), we know that for the large i , L_i has a unique eigenvalue μ_i satisfying $\mu_i \rightarrow 0$ and all the other eigenvalues of L_i have positive real parts and apart from 0.

Now we determine the sign of $\text{Re}(\mu_i)$ as $i \gg 1$. Let (ξ_i, η_i) be the corresponding eigenfunction to μ_i such that $(\xi_i, \eta_i) \rightarrow (\Phi, 0)$. Multiplying the first equation of $L_i(\xi_i, \eta_i) = \mu_i(\xi_i, \eta_i)$ by u_i and integrating the results over Ω , we obtain

$$-\int_{\Omega} u_i \Delta \xi_i dx - \int_{\Omega} u_i \xi_i \left(a_i - 2u_i - \frac{bv_i f(u_i, v_i)}{1 + \alpha u_i} \right) dx + \int_{\Omega} \frac{bu_i^2 f(u_i, v_i)}{1 + \beta v_i} \eta_i dx = \mu_i \int_{\Omega} u_i \xi_i dx. \quad (5.4)$$

By multiplying ξ_i to the first equation of (1.2) with $(u, v) = (u_i, v_i)$ and integrating the results over Ω , we have

$$-\int_{\Omega} u_i \Delta \xi_i dx = -\int_{\Omega} \xi_i \Delta u_i dx = \int_{\Omega} \xi_i u_i (a_i - u_i - bv_i f(u_i, v_i)) dx.$$

This combines with (5.4) yields

$$\mu_i \int_{\Omega} u_i \xi_i \, dx = \int_{\Omega} \xi_i u_i^2 \left(1 - \frac{b\alpha v_i f(u_i, v_i)}{1 + \alpha u_i} \right) dx + \int_{\Omega} \frac{bu_i^2 f(u_i, v_i)}{1 + \beta v_i} \eta_i \, dx. \quad (5.5)$$

Recall that $(u_i, v_i) = (s_i \Phi + O(s_i^2), v_c + s_i \Psi_{d_i} + O(s_i^2))$, and $\xi_i \rightarrow \Phi$, $\eta_i \rightarrow 0$, taking the real part in (5.5) firstly, then dividing the results by s_i^2 and letting $i \rightarrow \infty$ finally, we have

$$\lim_{i \rightarrow \infty} \frac{\operatorname{Re}(\mu_i)}{s_i} = \frac{\int_{\Omega} \Phi^3 (1 - \frac{\alpha b v_c}{1 + \beta v_c}) \, dx}{\int_{\Omega} \Phi^2 \, dx} \neq 0,$$

which implies $\operatorname{Re}(\mu_i) \neq 0$ for large i . Since all the other eigenvalues of L_i have positive real parts and apart from 0, the first conclusion is true.

Step 2. Now we prove the second conclusion. A contradiction argument will be used. Assuming that (1.2) has a unique positive solution (\hat{u}, \hat{v}) . Since $a_1 < 0$, $a < \tilde{a}$ and nears \tilde{a} , by the conclusion (i) of Theorem 9 we know that (\hat{u}, \hat{v}) must be the positive solution bifurcating from $(0, v_c, \tilde{a})$. That is, $(\hat{u}, \hat{v}) = (u(s), v(s))$. It is non-degenerate by the conclusion of Step 1. Therefore, $I - F'(u(s), v(s)): \overline{W}_{(u(s), v(s))} \rightarrow \overline{W}_{(u(s), v(s))}$ is invertible. Similar to the proof of Theorem 5, $F'(u(s), v(s))$ does not have property α . Consequently, $\operatorname{index}_W(F, (u(s), v(s))) = \pm 1$ by Proposition 1. Note that $\lambda_1 < a < \tilde{a} = \lambda_1(\frac{bv_c}{1+\beta v_c})$, applying Lemmas 1 and 2, we have that

$$\begin{aligned} 1 &= \deg_W(I - F, D) \\ &= \operatorname{index}_W(F, (0, 0)) + \operatorname{index}_W(F, (u_a, 0)) + \operatorname{index}_W(F, (0, v_c)) + \operatorname{index}_W(F, (\hat{u}, \hat{v})) \\ &= 0 + 0 + 1 \pm 1, \end{aligned}$$

which is a contradiction. This completes the proof. \square

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